AIT-BUDAPEST

AQUINCUM INSTITUTE OF TECHNOLOGY

Combinatorial Optimization

Sample Final Exam

Instructions

You have 100 minutes to work on the following problems. You may use a calculator but only for elementary calculations; that is, please do not use your calculator for graphing or carrying out programs.

Questions 1 to 6 are marked out of 10. Furthermore, you can collect bonus points by the two bonus problems; these are worth 5 points each. However, solving the bonus problems is optional: the $6 \times 10 = 60$ points achievable by questions 1 to 6 corresponds to a 100 % result.

Please make sure to justify your answers in detail. Please write down everything (including calculations, references to material covered, etc) that documents the process of your solution.

Please work in this booklet and only use extra pages if necessary.

1. Answer the following short theoretical questions.

a) Claim the Duality Theorem (in the form you wish).

b) The value of a certain two player, zero sum game is 10. What is the meaning of this information? Answer the question by interpreting this information from both players' point of view.

c) The system of linear inequalities $Ax \leq b$ consists of 30 inequalities. In 10 of these inequalities the coefficient of the first variable x_1 is 5, in another 10 inequalities the coefficient of x_1 is -3 and the remaining 10 inequalities do not contain x_1 at all. If the Fourier-Motzkin elimination is applied on this system, how many rows does the augmented coefficient matrix $(A|b)$ have after the first elimination step?

d) Define the Multicommodity Flow Problem (in terms of input, output and objective) and argue (very briefly) that it is solvable in polynomial time!

e) Define the notion of a cut and the capacity of a cut in a network. Argue that these notions are relevant in connection to the (single commodity) maximum flow problem and claim the corresponding theorem of Ford and Fulkerson.

2. Answer ONE of the following two questions.

2A. Claim and prove the (first form of the) Farkas-lemma. For the proof, you can rely on the Fourier-Motzkin elimination (and the fact that is works correctly).

(By the "first form" of the Farkas-lemma we mean the version that gives a necessary and sufficient condition for the solvability of a general system of linear inequalities.)

OR

2B. Claim and prove Egerváry's theorem on the maximum weight of a matching in a bipartite graph. For the proof, you can rely on the major theorem that deals with linear programs with TU coefficient matrices and the fact that the incidence matrices of bipartite graphs are TU.

3. a) Find the dual of the following linear program and reduce it to as simple a form as possible. (Give the dual in a form similar to that of the primal program; that is, do *not* use a matrix form. Note that no nonnegativity constraint is imposed on x_4.)

b) Determine the maximum value of the (primal) program.

$$\begin{aligned}
&\max\{x_1 + 6x_2 - x_4\}\\
&\text{subject to}\\
&2x_2 - 7x_3 - x_4 \leq -1\\
&2x_1 + 5x_3 + 3x_4 \geq 6\\
&7x_1 + 5x_2 - 4x_4 \leq 0\\
&x_1 + 3x_2 + 4x_3 = 1\\
&x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
\end{aligned}$$

4. The following linear program got injured in an unfortunate accident. $\square$'s represent all the data that was lost in the accident.

$$\begin{aligned}
&\max\{5x_1 + \square x_2 + 8x_3\}\\
&\text{subject to}\\
&x_1 + x_2 + 9x_3 \leq 5\\
&3x_1 + 4x_2 + 13x_3 \leq 12\\
&5x_1 + x_2 + 17x_3 \leq \square\\
&x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
\end{aligned}$$

Determine the maximum value of the program and restore the lost values if we know that $x_1 = 1$, $x_2 = 2$, $x_3 = 0$ is an optimal solution of the program.

5. Decide if the following operations preserve the TU property of any matrix.(In other words: is it true that applying one of the following operations on a TU matrix always yields another TU matrix?)

a) Adding a column with all its entries equal to 1.

b) Replacing each -1 entry by 1 and each 1 entry by -1.

c) Replacing one of its nonzero entries by 0.

6. Let $A = \{a_1, a_2, a_3, a_4\}$, $B = \{b_1, b_2, b_3, b_4\}$ and assume that a_i is adjacent to b_j in the bipartite graph $G(A, B; E)$ for every $1 \le i, j \le 4$. Furthermore, let the weight of the edge $\{a_i, b_j\}$ be the entry in the intersection of the i^{th} row and the j^{th} column of the following matrix for every $1 \le i, j \le 4$. Use the Egerváry-algorithm to find a maximum weight perfect matching in G!

$$\begin{pmatrix} 8 & 3 & 5 & 4 \\ 7 & 1 & 6 & 2 \\ 9 & 3 & 4 & 1 \\ 4 & 2 & 7 & 5 \end{pmatrix}$$

1. Bonus problem. Assume that x^* is an optimal solution of the linear program $\max\{cx : A_1x \leq b_1, A_2x \leq b_2\}$. Prove that the vector c can be broken up into the sum of the vectors $c = c_1 + c_2$ such that x^* is an optimal solution of both linear programs $\max\{c_1x : A_1x \leq b_1\}$ and $\max\{c_2x : A_2x \leq b_2\}$.

2. Bonus problem. A subset X of vertices in a directed graph G is called a *source* if no arc of G enters X and $X \neq V(G)$. Assume that a directed graph G is given together with a weight function $w : V(G) \to \mathbb{R}$ on its nodes. Prove that a source with maximum total weight of its nodes can be found in polynomial time!